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ASYMPTOTIC METHODS OF SOLVING THE SYSTEM OF KINETIC EQUATIONS OF A GASEOUS MIXTURE*

V.V. STRUMINSKII

The possibilities of the Chapman-Enskog method and a method proposed in this paper for solving the system of kinetic equations of a gaseous mixture are considered. It is shown that, for initial assumptions of the first order, the proposed method gives much more complete information on the system.

Methods of solving the system of kinetic equations of a gaseous mixtures are of considerable scientific and practical importance. The first approximations of the solution were obtained by Boltzmann himself. At the beginning of the present century, the Chapman-Enskog asymptotic method was developed to solve the kinetic Boltzmann equation. This method was then used to solve the system of kinetic equations for gaseous mixtures, where the important assumption was made that, in the zeroth approximation, the contribution of all the collision integrals to the change in the distribution functions of the individual components of the gas is zero. This is justified in extremely limited cases and only for a gas which is in a state close to a state of thermodynamic equilibrium. These assumptions led to a considerable reduction in the amount of information on the behaviour of the dynamic system and, in particular, on the motion of the individual components of gaseous mixtures.

In view of this, at the beginning of the Seventies we proposed a new, more-general, asymptotic method of solving the system of kinetic equations of a gaseous mixture, which enables the laws of motion of the individual components of the gaseous mixture to be determined for a system which is in a state quite far from a state of equilibrium, and then to determine the mean-mass parameters of this system as a whole. When developing this method important assumptions were also made regarding the contribution made by the different collision integrals to the change in the distribution functions.

In this paper we will consider both of these methods and the initial assumptions which underlie them. We will show that these assumptions are of the same order, although in the method proposed here more-complete information is obtained on the behaviour of the whole system. The problem of when it is best to use one or other method will also be considered.

The system of kinetic equations of a gaseous mixture in dimensionless variables can be written in the form

$$\frac{d\bar{f}_s}{dt} = \sum_{\tau=1}^{M} \frac{1}{\varepsilon_{s\tau}} \int\int\int_{\omega} (\bar{f}_s^{(1)}\bar{f}_\tau^{(1)} - \bar{f}_s\bar{f}_\tau)\, q_{s\tau} b db d\varepsilon d\mathbf{V}_\tau = \sum_{\tau=1} \frac{1}{\varepsilon_{s\tau}} J(\bar{f}_s\bar{f}_\tau) \tag{1}$$

where $\varepsilon_{s\tau}$ is the set of small parameters defining the collision frequency of the molecules, $\bar{f}_s$ are dimensionless functions of time, the coordinates and the parameters of the system, n_s is the number of particles per unit volume, m_s are the masses of the molecules, T_s are the temperatures and U_s are the mean volocities characterizing the given component.

The parameters $\varepsilon_{s\tau} \lesssim 1$ which occur in Eq.(1) and which satisfy the approximate equation $\varepsilon_{s\tau} = l/\lambda_{s\tau}$, are purely tentative. The true contribution of the collision integrals to the change in $d\bar{f}_s/dt$ is determined by the distribution functions and the laws of interaction of the molecules themselves.

According to the Chapman-Enskog method /1/, the allowable state of a gaseous mixture is such that all the $\varepsilon_{s\tau}$ are approximately equal to a small parameter ε. In this case, we obtain from Eq.(1)

$$\varepsilon \frac{d\bar{f}_s}{dt} = \sum_{\tau=1}^{M} J(\bar{f}_s\bar{f}_\tau) \tag{2}$$

Assuming, as is usually done, that

$$\bar{f}_s = \bar{f}_s^{(0)} + \varepsilon\bar{f}_s^{(1)} + \varepsilon\bar{f}_s^{(2)} + \ldots \tag{3}$$

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we obtain

$$\sum_{\tau=1}^{M} J(\bar{f}_s^{(0)}\bar{f}_\tau^{(0)})=0, \quad \frac{d\bar{f}_s^{(0)}}{dt}=\sum_{\tau=1}^{M} J(\bar{f}_s^{(0)}\bar{f}_\tau^{(1)})+J(\bar{f}_s^{(1)}\bar{f}_\tau^{(0)}) \tag{4}$$

$$\frac{d\bar{f}_s^{(1)}}{dt}=\sum_{\tau=1}^{M} J(\bar{f}_s^{(0)}\bar{f}_\tau^{(2)})+J(\bar{f}_\tau^{(0)}\bar{f}_s^{(2)})+J(\bar{f}_s^{(1)}\bar{f}_\tau^{(1)}),\ldots$$

In the zeroth approximation, all the $\bar{f}_s^{(0)}$ are locally Maxwell functions, defined by the mean-mass parameters of the flow U_0, T_0, n_s and m_s, and the contribution of collision integrals to the change in $d\bar{f}_s^{(0)}/dt$ is zero. We obtain the Navier-Stokes, Barnett etc. systems of equations for determining the mean-mass parameters of the flow from the conditions for the inhomogeneous integral equations to be solvable. To a first approximation, in the Navier-Stokes systems of equations completely defined values of the viscosity and thermal conductivity are obtained. Of course, for the assumptions used in the method these values of the coefficients may not be completely accurate, but can be refined in subsequent approximations.

The following approximation was calculated in /2/ using certain assumptions. It was shown that in addition to the purely Barnett (non-linear) terms, terms also appear in this approximation that are linear with respect to the parameter gradients, which introduce corrections to the transfer coefficients. The value of these corrections will be determined by the state of the gaseous system. The more accurately the assumptions made in the method are satisfied in the system, the closer it will be to the state of equilibrium, and the less the higher approximations that will be required and the smaller will be the value of the corrections.

As can be seen, the Chapman-Enskog method only determines the mean-mass parameters of the flow, which is in a state close to equilibrium. It says absolutely nothing about the law of motion of the individual components of the gaseous mixture.

In the method proposed in /3/, it was assumed that the state of the gaseous mixture deviates quite considerably from a state of equilibrium and such that we can assume that when $\tau = s$, the value of $\varepsilon_{s\tau}$ is equal to the small parameter ε, and for all the remaining ones $\varepsilon_{s\tau} \leqslant 1$. In this case, system (1) can be written, in particular, in the form

$$\varepsilon\left(\frac{d\bar{f}_s}{dt}-\sum_{\tau\neq s}^{M} J(\bar{f}_s\bar{f}_\tau)\right)=J(\bar{f}_s\bar{f}_s) \tag{5}$$

After making fundamental assumptions both in the Chapman-Enskog method and in the method described in /3/, we obtain system (5) from system (1), which is already quite sufficient (as was stated repeatedly in /4/) to obtain all further information on the gas-dynamic behaviour of the system.

From (3) and (5) we also obtain the following recurrent system of equations:

$$J(\bar{f}_s^{(0)}\bar{f}_s^{(0)})=0, \quad \left(\frac{d\bar{f}_s^{(0)}}{dt}-\sum_{\tau\neq s}^{M} J(\bar{f}_s^{(0)}\bar{f}_\tau^{(0)})\right)=2J(\bar{f}_s^{(0)}\bar{f}_s^{(1)}) \tag{6}$$

$$\frac{df_s^{(1)}}{dt}-\sum_{\tau\neq s}^{M}(J(\bar{f}_s^{(1)}\bar{f}_\tau^{(0)})+J(\bar{f}_s^{(0)}\bar{f}_\tau^{(1)}))=2J(\bar{f}_s^{(0)}\bar{f}_\tau^{(2)})+J(\bar{f}_s^{(0)}\bar{f}_s^{(1)})+\ldots,\ldots$$

In the zeroth approximation, the function $\bar{f}_s^{(0)}$ will also be locally Maxwellian, but now it is defined by the parameters U_s, T_s and n_s, characerizing the given component of the gas. We obtain from the first inhomogeneous equation of the system of Eqs.(6) the following gas-dynamic system of equations:

$$\frac{\partial U_s^\alpha}{\partial t}+U_s^\beta\frac{\partial U_s^\alpha}{\partial r_\beta}+\frac{1}{\rho_s}\frac{\partial \vec{P}_s^{\alpha\beta}}{\partial r_\beta}+\frac{1}{\rho_s}\frac{\partial P_s}{\partial r_\alpha}=\frac{16}{3\rho_s}\sum_{\tau\neq s}^{M}\frac{\rho_s\rho_\tau\Omega_{s\tau}^{(1,1)}}{m_\tau+m_s}(U_\tau^\alpha-U_s^\alpha) \tag{7}$$

$$\frac{3}{2}n_s k\left(\frac{\partial T_s}{\partial t}+U_s^\alpha\frac{\partial T_s}{\partial r_\alpha}\right)+P_s\frac{\partial U_s^\alpha}{\partial r_\alpha}+\vec{P}_s^{\alpha\beta}\frac{\partial U_s^\alpha}{\partial r_\beta}+\frac{\partial \vec{q}_s^{\alpha}}{\partial r_\alpha}=$$

$$16k\sum_{\tau\neq s}^{M}\frac{\rho_s\rho_\tau\Omega_{s\tau}^{(1,1)}}{(m_\tau+m_s)^2}((T_\tau-T_s)+\omega_s(U_\tau-U_s)^2)$$

$$\frac{\partial\rho_s}{\partial t}+\frac{\partial U_s^\alpha\rho_s}{\partial r_\alpha}=0$$

where ω_s and $\Omega_{s\tau}$ are well-known expressions and integrals.

This generalized system of Navier-Stokes equations emphasizes one again that the structure of the classical system of Navier-Stokes equations is uniquely determined by the assumptions made in deriving (2). Exactly the same structure of the generalized system of Navier-Stokes Eqs.(7) is uniquely defined by the assumptions made in deriving (5). In both cases, on changing from one gaseous system to another, close to the initial one, only the transfer coefficients may be changed. Here one must bear in mind the following: the better the gaseous system satisfies the initial assumptions, the more accurately the transfer coefficients in the first approximation will correspond to reality. The need to resort to higher approximations will be unnecessary, although there is always the possibility of refining the transfer coefficient using higher approximations.

For the Chapman-Enskog method, this possibility was proved in /2/. For our method, it was also shown in /5/ that higher approximations contain terms that are linear in the gradients of the flow parameters, and their values were approximately determined.

Hence, the kinetic equations for a gaseous mixture are far more complex than for a single-component gas. In a gaseous mixture there are an enormous number of quite different states.

A comparatively small number of states of gaseous mixtures, close to the state of thermodynamic equilibrium, can be described, to a first approximation, by the Chapman-Enskog method. To a second approximation, the structure of the gas-dynamic Navier-Stokes equations remains unchanged, if we ignore the Bernett terms, and the transfer coefficient can be refined. Unfortunately, the laws of motion of the components of the gaseous mixture cannot be determined by this method.

A large number of gaseous mixtures, not in a state of thermodynamic equilibrium, can be described, to a first approximation, by our method. The system of Navier-Stokes equations, which we have generalized, is not changed in either the second or subsequent approximations if we ignore the quadratic Barnett terms. Only the transfer coefficients can be refined in subsequent approximations.

We will consider a number of specific examples which illustrate the possibilities of our method.

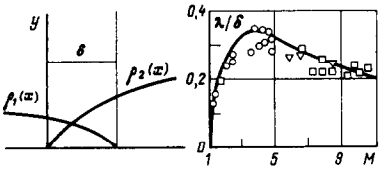

Fig.1

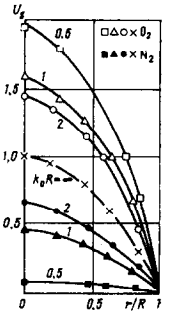

Fig.2

We will use the new method to describe the structure of intense shock waves. We note first of all that a number of papers have been published on the construction of the theory of shock waves using the Navier-Stokes equations, in which agreement with experiment was only obtained for low Mach numbers. Beginning with the papers by Mott-Smith, the initial kinetic equation was used in the theory with additional assumptions and empirical data. But this also does not enable one to consider the region of high velocities. We have attempted to provide a further generalization of the kinetic theory in a number of papers.

By dispensing with the assumption that the distribution function is symmetrical /6, 7/, to describe individual groups of molecules we obtained a system of kinetic equations similar to the system of equations for gaseous mixtures. The method described above was applied to this system, and when only the diffusion terms were taken into account, ignoring the viscosity and thermal conductivity of the gas (the zeroth approximation), we were able to obtain a solution in analytical form.

Theory and experiment are compared in Fig.1. The Prandtl thickness of the shockwave is given by the expression $\delta = (\rho_+ - \rho_-)/(dp/dx)_{\max}$, where M is the Mach number. We used the most reliable experiments for argon at a temperature of 300°C.

We will consider, as the second example, the flow of a binary gaseous mixture in thin channels and capillary tubes. Starting from (7), we obtain the following system of two mutually connected ordinary differential equations:

$$\frac{d^2U_1}{dx^2} + \frac{d}{\mu}(U_1 - U_2) = \frac{1}{\mu_1}\frac{dp_1}{dx}, \quad \frac{d^2U_2}{dx^2} + \frac{d}{\mu}(U_2 - U_1) = \frac{1}{\mu_2}\frac{dp_2}{dx} \tag{8}$$

Here

$$k^2 = d\left(\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right), \quad \omega_s^2 = \frac{1}{\mu_s}\frac{\partial p_s}{\partial x} - \frac{1}{\mu_1+\mu_2}\left(\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x}\right)$$

The accurate solution will have the form

$$U_s = \frac{1}{\mu_1+\mu_2}\left(\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x}\right) + \frac{\omega^2}{k_0^2}\left(\frac{\operatorname{ch} ky}{\operatorname{ch} kh} - 1\right) \tag{9}$$

and shows that the effect is largely determined by the intensity of the interaction of the components of the gas, which is characterized by the parameter k_0R (Fig. 2). In the case of weak interaction, when the value of this parameter is small ($k_0R = 0.5, 1$ and 2), the components of the binary mixture behave as though they are independent. It is interesting to note that in the case of strong interaction ($k_0R = \infty$), the flow again acquires the form of Poiseuille flow with the overall viscosity of the components and with the overall gradient.

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EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS*

V.I. GRYN

Considering steady Hiemenz-Birman flows only, a study is made of flows between porous walls, on the assumption that fluid is injected and extracted at identical rates. It is shown that wherever fluid is being extracted a boundary layer forms at the wall. A class of unsteady two-dimensional flows, more general than Hiemenz-Birman flow, is investigated. In a class of flows generalized Jeffrey-Hamel flow, attention is devoted to flows in a dihedral angle between porous walls when fluid is injected and extracted. A class of steady (unsteady) two-dimensional flows is found, in which flow between coaxial porous cylinders, with fluid injected and extracted at arbitrary rates, is considered. Some exact solutions of the steady- and unsteady-state Navier-Stokes equations are found.

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